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On the Existence, Uniqueness and Stability of Solutions of a Nonlinear Functional Differential Equation*

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1. INTRODUCTION

Let R be the reals and R^+ the positive reals. If g maps R into R, $g^t(\cdot)$, the history of g to time t, is a map from R^+ into R defined by $g^t(s) = g(t - s)$. Let H be some Banach space of functions each mapping R^+ into R and let σ be a nonlinear functional mapping $R \times H$ into R. In this paper we study the nonlinear functional differential equation

$$u_{tt}(x, t) = \frac{\partial}{\partial x} \sigma(u_x(x, t), u_x{}^t(x, \cdot)) + \lambda u_{xtx}(x, t) + f(x, t) \tag{1.1}$$

together with

$$\begin{aligned} u(x, t) &= u_0(x, t), & (x, t) &\in [0, \pi] \times [-\infty, 0], \\ \lim_{t\to 0^+} u_t(x, t) &= u_1(x), & x &\in [0, \pi], \end{aligned} \tag{1.2}$$

and

$$u(0, t) = u(\pi, t) = 0, \qquad t > 0, \tag{1.3}$$

when σ is what we call *dissipative* and *stable*. The initial-boundary value problem consisting of (1.1), (1.2) and (1.3) will be referred to as problem (P_λ).

By a solution of (P_λ), $\lambda > 0$, we mean a function u defined on $[0, \pi] \times (-\infty, \infty)$ satisfying

(1) $u \in C^2((0, \pi) \times (0, \infty))$,

(2) $u_{xtx} \in C^0((0, \pi) \times (0, \infty))$,

(3) all of the indicated derivative in (1) and (2) have continuous extensions to $[0, \pi] \times (0, \infty)$, and

(4) (1.1), (1.2) and (1.3) hold.

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When $\lambda = 0$, (1.1) is the balance of momentum law for a one-dimensional material with memory. It is believed that solutions of problems (P_λ), $\lambda > 0$, converge to the solution of problem (P_0).

To simplify some of the calculations, we will assume that u_0 and u_1 are identically zero. We will further assume that H is a weighted L_2 space. More precisely, let $h \in L_1(R^+) \cap C([0, \infty))$ be nonnegative and bounded. Then

$$H = \left\{ g : R^+ \to R \mid g \text{ is measurable and } |g|_H^2 = \int_0^\infty h(t) g^2(t)\, dt < \infty \right\}.$$

With the usual identification of functions, H is a Hilbert space. (We could have assumed, with only minor modifications of the ensuing analysis, that H is a weighted L_p space, $p \geqslant 1$). Assumptions on f will be stated later (see formula 4.1).

In the next section the notion of dissipative and stable functionals are introduced. In Section 3 the assumptions on σ are listed and discussed. Theorem 4.1 is a statement of certain a priori estimates; the existence and uniqueness theorem is also discussed in Section 4. The asymptotic stability of u_t is proven in Section 5. In another paper we shall discuss the asymptotic behavior of the solution under an additional hypothesis on σ.

2. Dissipative and Stable Functionals

Let

$$A^i = \{g \in C^i(R) \mid g^t \in H \text{ for every } t \in R^+\}$$

and

$$A_0{}^i = \{g \in A^i \mid \text{there exists } t_0(g) \in R \text{ such that } g(t) = 0 \text{ when } t \leqslant t_0(g)\}.$$

Definition 2.1. $\sigma : R \times H \to R$ is *dissipative* if there exists $k > 0$ such that for every $g \in A_0{}^1$ and $t_1 > t_0(g)$,

$$\int_{-\infty}^{t_1} \dot{g}(t)\, \sigma(g(t), g^t)\, dt \geqslant \frac{k}{2} g^2(t_1). \tag{2.1}$$

(We will call k the dissipative constant.)

The following easily proven proposition gives a sufficient condition for (2.1) in terms of the Frechet derivative of σ. By the Frechet derivative of σ at $(\zeta, f) \in R \times H$ we mean a continuous linear functional $L_0(\zeta, f \mid \cdot) : H \to R$ satisfying

$$\sigma(\zeta, f + g) = \sigma(\zeta, f) + L_0(\zeta, f \mid g) + o(|g|_H).$$

In all that follows, we will assume that the Frechet derivative of σ is continuous when viewed as a map from $R \times H \times H$ into R.

PROPOSITION 2.1. *If*

1. $\sigma_\zeta(\zeta, 0) \geqslant m_0 > 0$ *for every* $\zeta \in R$,
2. $\sigma(0, 0) = 0$,

and

3. *there exists* $k > 0$ *such that for every* $(g, \varphi) \in A_0^1 \times A_0^1$ *and* $t_1 > t_0(g)$,

$$\frac{m_0}{2} g^2(t_1) + \int_{-\infty}^{t_1} \dot{g}(t) L_0(g(t), \varphi^t \mid g^t)\, dt \geqslant \frac{k}{2} g^2(t_1), \tag{2.2}$$

then σ is dissipative with dissipative constant k.

The concept of a stable functional is best motivated by considering the linear zero dimensional theory of materials with memory. In this theory

$$\sigma(\zeta, f) = m_0 \zeta + Lf \tag{2.3}$$

where $m_0 > 0$ and L is a continuous linear functional on H. By the Riesz Theorem, there exists $\Gamma \in H$ such that, for every $f \in H$,

$$Lf = \int_0^\infty h(\tau)\, \Gamma(\tau) f(\tau)\, d\tau. \tag{2.4}$$

The classical notation of viscoelasticity can be achieved if G is defined by

$$G(\tau) = m_0 + \int_0^\tau h(s)\, \Gamma(s)\, ds. \tag{2.5}$$

Then (2.3) becomes

$$\sigma(\zeta, f) = G(0)\, \zeta + \int_0^\infty G'(\tau) f(\tau)\, d\tau. \tag{2.6}$$

It is shown in the linear theory of [4] that the sign of $G(\infty)$ determines stability. In that theory $G' = h\Gamma$ was assumed to be negative, and the stability criterion can be written as

$$G(\infty) = m_0 + \int_0^\infty h(s)\, \Gamma(s)\, ds = m_0 - \int_0^\infty | h(s)\, \Gamma(s)|\, ds > 0.$$

The analog of this inequality for a nonlinear functional σ will be the stable functional criterion.

By the Riesz theorem there exists $\Gamma(\zeta, f; \cdot) \in H$ such that for every $(\zeta, f, g) \in R \times H \times H$,

$$L_0(\zeta, f \mid g) = \int_0^\infty h(\tau)\, \Gamma(\zeta, f; \tau)\, g(\tau)\, d\tau.$$

Assume that $\sigma_\zeta(\zeta, 0) \geqslant m_0 > 0$ for every $\zeta \in R$.

DEFINITION 2.2. $\sigma : R \times H \to R$ is stable if there exists $\mathscr{R} \in (0, 1)$ such that, for every $(\zeta, f) \in R \times H$,

$$\mathscr{R} m_0 - \int_0^\infty | h(\tau)\, \Gamma(\zeta, f; \tau)|\, d\tau > 0. \tag{2.7}$$

We now exhibit a class of functionals each of which is both dissipative and stable. These functionals will be nonlinear in the present and linear in the past. We will say that a function $K \in C([0, \infty))$ is of positive type if

$$\int_{t_0}^{t_1} f(t) \int_{t_0}^{t} K(t - \tau) f(\tau)\, d\tau\, dt \geqslant 0$$

for every real t_0 and t_1, $t_0 < t_1$, and every $f \in C(R)$. The relationship between this definition and the classical one is discussed in [3]. The notation we use is consistent with formulas (2.4) and (2.5).

THEOREM 2.2. *If*

1. $\hat{\sigma}'(\zeta) \geqslant m_0 > 0$ *for every* $\zeta \in R$, *and* $\hat{\sigma}(0) = 0$,
2. $h(\tau)\, \Gamma(\tau) \in C([0, \infty))$ *and is nonpositive,*
3. $G(\infty) > 0$ (*G is defined by* (2.5)),

and

4. $\tilde{G}(t) = G(t) - G(\infty)$ *is of positive type,*

then

$$\sigma(\zeta, f) = \hat{\sigma}(\zeta) + \int_0^\infty G'(\tau) f(\tau)\, d\tau$$

is stable and dissipative with dissipative constant $G(\infty)$.

Proof. That σ is stable follows from 2 and 3. To show that σ is dissipative, it suffices to show (2.2). We have $m_0 = G(0)$, and

$$L_0(\zeta, f \mid g) = \int_0^\infty \tilde{G}'(\tau)\, g(\tau)\, d\tau,$$

and by integrating by parts and using 4,

$$\frac{G(0)}{2}g^2(t_1)+\int_{-\infty}^{t_1}\dot{g}(t)\int_0^\infty \tilde{G}'(\tau)\,g(t-\tau)\,d\tau\,dt$$

$$=\frac{G(\infty)}{2}g^2(t_1)+\int_{-\infty}^{t_1}\dot{g}(t)\int_{-\infty}^{t}\tilde{G}(t-\tau)\,\dot{g}(\tau)\,d\tau\,dt$$

$$\geqslant\frac{G(\infty)}{2}g^2(t_1).$$

3. Assumptions on σ

We now list and comment on the assumptions made concerning σ. Recall that $L_0(\zeta, f \mid g)$ is the Frechet derivative of σ at (ζ, f) evaluated at g and, by the Riesz theorem,

$$L_0(\zeta, f \mid g)=\int_0^\infty h(\tau)\,\Gamma(\zeta, f;\tau)\,g(\tau)\,d\tau.$$

(A.1) σ is dissipative and stable.

(A.2) $\sigma(0,0)=0$

(A.3) $\sigma_\zeta : R\times H\to R$ is continuous and

$$0<m_0=\inf_{(\zeta,f)\in R\times H}\sigma_\zeta(\zeta,f)\leqslant\sup_{(\zeta,f)\in R\times H}\sigma_\zeta(\zeta,f)=M_0<\infty.$$

The upper bound on σ_ζ is not physically reasonable. It is possible that σ_ζ approaches $+\infty$ as ζ approaches -1 from above. Because of the a priori estimates of Theorem 4.1, this possibility can be included in a standard manner if we require $u_x \geqslant \delta - 1$, $0<\delta<1$. (Positivity of density implies $u_x > -1$.)

(A.4) For every $T>0$,

$$\sup_{\substack{(\zeta,f)\in R\times H\\ t\in[0,T]}}|\,\Gamma(\zeta,f;t)|<\infty.$$

Hypothesis (A.4) appears to be needed for the existence proof. (See [1] for details.) Let $L_1(\zeta, f \mid g)$ be the Frechet derivative of σ_ζ at $(\zeta, f)\in R\times H$ evaluated at g and $L_2(\zeta, f, \tau \mid g)$ the Frechet derivative of Γ at

$$(\zeta, f, \tau)\in R\times H\times R^+$$

evaluated at g.

(A.5) For every $g \in H$, there exists $N(g) > 0$ such that for every $(\zeta, f, t) \in R \times H \times R^+$

$$|L_i(\zeta, f \mid g)| \leqslant N(g), \qquad i = 0, 1,$$

and

$$|L_2(\zeta, f, t \mid g)| \leqslant N(g).$$

By the uniform boundedness principle and (A.5), there exists $N_i > 0$, $i = 0, 1, 2$, such that

$$\sup_{\substack{|g|_H=1 \\ (\zeta,f)\in R\times H}} |L_i(\zeta, f \mid g)| \leqslant N_i, \qquad i = 0, 1, \tag{3.1}$$

and

$$\sup_{\substack{|g|_H=1 \\ (\zeta,f,\tau)\in R\times H\times R^+}} |L_2(\zeta, f, \tau \mid g)| \leqslant N_2. \tag{3.2}$$

In what follows we require additional differentiability and that certain functionals map balls into bounded sets. These further technical assumptions will not be commented on. We conclude this section with two lemmas; the first is well known, the second is straightforward.

LEMMA 3.1. *If $\rho : H \to R$ is Frechet differentiable (with derivative L), then for every $(f, g) \in H \times H$, there exists $\zeta \in (0, 1)$ such that*

$$\rho(f + g) = \rho(f) + L(f + \zeta g \mid g).$$

LEMMA 3.2. *If*

$$\sup_{t\in R} \int_0^\pi f^2(x, t)\, dx < \infty,$$

then

$$\int_0^\pi \sigma^2(f(x, t), f^t(x, \cdot))\, dx \leqslant M_1 \sup_{t\in R} \int_0^\pi f^2(x, t)\, dx$$

where

$$M_1 = 2\left(M_0^2 + N_0^2 \int_0^\infty h(\tau)\, d\tau\right).$$

4. A Priori Estimates, Existence and Uniqueness

An outline of the ideas of this section is as follows: First we obtain energy estimates for various derivatives of u. These enable us to obtain, by Fourier series methods, a priori pointwise bounds for u_t. We then "solve" the equation for u_{xx} in terms of u_t. This enables us to obtain pointwise estimates for u_{xx}. We are then able to then prove the existence and uniqueness theorem. Let

$$|g|_\infty = \sup_{p\in\text{Domain of } g} |g(p)|,$$

$$|g|_{t,\infty} = \sup_{(x,\tau)\in[0,\pi]\times[t,\infty)} |g(x,\tau)|,$$

$$|\varphi|(t) = \sup_{x\in[0,\pi]} |\varphi(x,t)|,$$

and

$$\|\varphi\|(t) = \left(\int_0^\pi \varphi^2(x,t)\,dx\right)^{1/2}.$$

In all that follows we assume that f, the forcing term of (1.1), satisfies

$$\begin{aligned}
&\text{(a)}\quad f\in C^2([0,\pi]\times[0,\infty));\\
&\text{(b)}\quad |f|_\infty < \infty;\\
&\text{(c)}\quad \int_0^\infty\int_0^\pi f^2(x,t)\,dx\,dt < \infty;\\
&\text{(d)}\quad f(0,t) = f(\pi,t) = 0, \quad t\geqslant 0.
\end{aligned} \tag{4.1}$$

The main result of this section is the following theorem, the proof of which is preceded by a sequence of lemmas all of which have the hypotheses of the theorem.

THEOREM 4.1. *If f satisfies* (4.1) *and u is a solution of problem* (P_λ), $\lambda > 0$, *then there exists* $C>0$, $C = C(1/\lambda, k, M_0, M_1, N_1, |h|_\infty, \mathscr{R})$ *such that*

$$\max_{t\in R^+}(\|u\|(t), |u|(t), \|u_x\|(t), |u_x|(t), \|u_t\|(t), |u_t|(t), \|u_{xx}\|(t), |u_{xx}|(t))$$
$$\leqslant C\left\{|f|_\infty + \left(\int_0^\infty\int_0^\pi f^2\right)^{1/2}\right\}.$$

LEMMA 4.1.

$$\left(\int_0^T\int_0^\pi u_{tx}^2\right)^{1/2} \leqslant \frac{\pi}{\lambda\sqrt{2}}\left(\int_0^T\int_0^\pi f^2\right)^{1/2}; \tag{4.2}$$

$$\|u_t\|^2(T) \leqslant \frac{\pi^2}{\lambda}\int_0^T\int_0^\pi f^2; \tag{4.3}$$

$$\|u_x\|^2(T) \leqslant \frac{\pi^2}{k\lambda}\int_0^T\int_0^\pi f^2. \tag{4.4}$$

Proof. Multiplying (1.1) by u_t, integrating over $[0, \pi] \times [0, T]$, integrating by parts and using (1.2) and (1.3), we have

$$\frac{1}{2}\int_0^\pi u_t^2(x, T)\,dx + \int_0^\pi\int_0^T u_{tx}\sigma(u_x, u_x^t)\,dt\,dx + \lambda\int_0^\pi\int_0^T u_{tx}^2\,dt\,dx = \int_0^\pi\int_0^T u_t f\,dt\,dx. \tag{4.5}$$

Integrating by parts and using the Schwarz inequality twice, we get

$$\int_0^T\int_0^\pi u_t f\,dx\,dt \leqslant \frac{\pi}{\sqrt{2}}\left(\int_0^T\int_0^\pi f^2\,dx\,dt\right)^{1/2}\left(\int_0^T\int_0^\pi u_{tx}^2\,dx\,dt\right)^{1/2}.$$

By the dissipative hypothesis and the above estimate, (4.5) yields

$$\frac{1}{2}\int_0^\pi u_t^2(x, T)\,dx + \frac{k}{2}\int_0^\pi u_x^2(x, T)\,dx + \lambda\int_0^\pi\int_0^T u_{tx}^2\,dt\,dx \leqslant \frac{\pi}{\sqrt{2}}\left(\int_0^T\int_0^\pi f^2\,dx\,dt\right)^{1/2}\left(\int_0^\pi\int_0^T u_{tx}^2\,dt\,dx\right)^{1/2}.$$

Thus,

$$\left(\int_0^T\int_0^\pi u_{tx}^2(x, t)\,dx\,dt\right)^{1/2} \leqslant \frac{\pi}{\lambda\sqrt{2}}\left(\int_0^T\int_0^\pi f^2\,dx\,dt\right)^{1/2}.$$

The last two inequalities imply

$$\int_0^\pi u_t^2(x, T)\,dx \leqslant \frac{\pi^2}{\lambda}\int_0^T\int_0^\pi f^2$$

and

$$\int_0^\pi u_x^2(x, T)\,dx \leqslant \frac{\pi^2}{k\lambda}\int_0^T\int_0^\pi f^2.$$

LEMMA 4.2.

$$|u_t|_{0,\infty} \leqslant C_1\left(\int_0^\infty\int_0^\pi f^2\right)^{1/2} + C_2\,|f|_\infty,$$

where

$$C_1 = C_1\left(\frac{1}{\lambda}, k, M_0, M_1, N_0, |h|_\infty\right) \qquad \textit{and} \qquad C_2 = C_2\left(\frac{1}{\lambda}\right).$$

Proof. Since

$$|u_t(x, t)| = \left|\int_0^x u_{tx}(\zeta, t)\,d\zeta\right| \leqslant \sqrt{\pi}\left(\int_0^\pi u_{tx}^2(x, t)\,dx\right)^{1/2},$$

it suffices to estimate $(\int_0^\pi u_{tx}^2(x, t)\,dx)^{1/2}$.

Consider the Fourier sine series expansions of the left and right sides of (1.1). Because of (1.1),

$$\int_0^\pi \sin nx\, u_{tt}(x,t)\,dx = \int_0^\pi \sin nx \left[\frac{\partial}{\partial x}\sigma(u_x(x,t), u_x{}^t(x,\cdot)) + \lambda u_{xtx}(x,t) + f(x,t)\right] dx, \qquad n = 1, 2,\ldots .$$

Integrating the term involving u_{xtx} by parts twice yields

$$\int_0^\pi \sin nx\ u_{tt}(x,t)\,dx + \lambda n^2 \int_0^\pi \sin nx u_t(x,t)\,dx = F_n(t) \tag{4.6}$$

where

$$F_n(t) = \int_0^\pi \sin nx \left[\frac{\partial}{\partial x}\sigma(u_x(x,t), u_x{}^t(x,\cdot)) + f(x,t)\right] dx.$$

Since (4.6) is an ordinary differential equation in

$$a_n(t) = \int_0^\pi \sin nx\ u_t(x,t)\,dx$$

and $a_n(0) = 0$, we have

$$\begin{aligned} a_n(T) &= \int_0^T \exp[-\lambda n^2(T-t)]\,F_n(t)\,dt \\ &= f_n(T) + \sigma_n(T) \end{aligned}$$

where

$$f_n(T) = \int_0^T \exp[-\lambda n^2(T-t)] \int_0^\pi \sin nx\ f(x,t)\,dx\,dt$$

and

$$\sigma_n(T) = \int_0^T \exp[-\lambda n^2(T-t)] \int_0^\pi \sin nx \frac{\partial}{\partial x}\sigma(u_x, u_x{}^t)\,dx\,dt.$$

Integrations by parts yield

$$\sigma_n(T) = \rho_n(T) + \mu_n(T) + \varphi_n(T),$$

where

$$\rho_n(T) = -\frac{1}{\lambda n}\int_0^\pi \cos nx\ \sigma(u_x, u_x{}^T)\,dx,$$

$$\mu_n(T) = \frac{1}{\lambda n}\int_0^\pi \cos nx \int_0^T \exp[-\lambda n^2(T-t)]\ \sigma_\xi(u_x, u_x{}^t)\,u_{xt}\,dt\,dx,$$

and

$$\varphi_n(T) = \frac{1}{\lambda n}\int_0^\pi \cos nx \int_0^T \exp[-\lambda n^2(T-t)]\,L_0(u_x, u_x{}^t \mid u_{xt}^t)\,dt\,dx.$$

Note that

$$u_{tx}(x, t) = \sum_{n=1}^{\infty} \frac{2}{\pi} \left(\int_0^{\pi} \cos nx \, u_{tx}(x, t) \, dx \right) \cos nx + \frac{1}{\pi} \int_0^{\pi} u_{tx}(x, t) \, dx$$

$$= - \sum_{n=1}^{\infty} \frac{2n}{\pi} \left(\int_0^{\pi} \sin nx \ u_t(x, t) \, dx \right) \cos nx$$

because of (1.3). Hence, we have

$$u_{tx}(x, t) = - \sum_{n=1}^{\infty} \frac{2n}{\pi} a_n(t) \cos nx$$

$$= - \frac{2}{\pi} \sum_{n=1}^{\infty} n(f_n(t) + \rho_n(t) + \mu_n(t) + \varphi_n(t)) \cos nx.$$

Furthermore,

$$\frac{1}{\lambda} \sigma(u_x(x, t), u_x{}^t(x, \cdot)) = \frac{1}{\lambda\pi} \int_0^{\pi} \sigma(u_x(x, t), u_x{}^t(x, \cdot)) \, dx - \frac{2}{\pi} \sum_{n=1}^{\infty} n\rho_n(t) \cos nx.$$

Hence,

$$u_{tx}(x, t) - \frac{1}{\lambda} \sigma(u_x(x, t), u_x{}^t(x, \cdot)) + \frac{1}{\lambda\pi} \int_0^{\pi} \sigma(u_x(x, t), u_x{}^t(x, \cdot)) \, dx$$

$$= - \frac{2}{\pi} \sum_{n=1}^{\infty} n(f_n(t) + \mu_n(t) + \varphi_n(t)) \cos nx.$$

Consequently,

$$\left(\int_0^{\pi} u_{tx}^2(x, t) \, dx \right)^{1/2} \leqslant \frac{2}{\lambda} \left(\int_0^{\pi} \sigma^2(u_x(x, t), u_x{}^t(x, \cdot)) \, dx \right)^{1/2} + \frac{2}{\pi} \left(\sum_{n=1}^{\infty} [n^2 f_n{}^2(t) + n^2 \mu_n{}^2(t) + n^2 \varphi_n{}^2(t)] \right)^{1/2}. \tag{4.7}$$

We now estimate each of the terms on the right side of (4.7): By Lemma 3.2, (4.4) and (4.1.c),

$$\left(\int_0^{\pi} \sigma^2(u_x , u_x{}^t) \, dx \right)^{1/2} \leqslant C_0 \left(\int_0^{\infty} \int_0^{\pi} f^2 \, dx \, dt \right)^{1/2}.$$

By (A.3), the Schwarz inequality and (4.2),

$$| n\mu_n(t)| \leqslant \frac{C_0}{n} \left(\int_0^{\infty} \int_0^{\pi} f^2 \, dx \, dt \right)^{1/2}.$$

Using 3.1, the Schwarz inequality and (4.2), we have

$$| n\varphi_n(t)| \leqslant \frac{C_0}{n^2}\left(\int_0^\infty\int_0^\pi f^2\,dx\,dt\right)^{1/2}.$$

C_0 in the above estimates depends on M_0, M_1, N_0, k, $| h|_\infty$ and $1/\lambda$. We also have $| nf_n(t)| \leqslant (1/\lambda n)| f|_\infty \pi$. Because of (4.7) and the preceding estimates, the lemma follows.

LEMMA 4.3.

$$| u_{xx}|_{0,\infty} \leqslant C_3(| f|_{0,\infty} + | u_t|_{0,\infty})$$

where

$$C_3 = C_3\left(\frac{1}{\lambda}, M_0, \mathscr{R}\right).$$

Proof. Using the Riesz theorem, we can write (1.1) as

$$\begin{aligned} &u_{xtx} + \frac{1}{\lambda}\,\sigma_\zeta(u_x, u_x{}^t)\,u_{xx} \\ &\quad = \frac{1}{\lambda}\left[u_{tt} - f - \int_0^t h(t-t)\,\Gamma(u_x, u_x{}^t; t-\tau)\,u_{xx}(x, t-\tau)\,d\tau\right]. \end{aligned} \tag{4.8}$$

Proceeding in a manner analogous to the derivation of (5.1) of [2], (4.8) leads to a Volterra type integral equation for u_{xx} in which the rightside depends on f and, after an integration by parts, u_t. The kernel of the Volterra operator is

$$\begin{aligned} &L(x, t_2, \tau) \\ &\quad = -\frac{1}{\lambda}\int_\tau^{t_2} \exp\left(-\frac{1}{\lambda}\int_t^{t_2}\sigma_\zeta(u_x, u_x{}^s)\,ds\right) h(t-\tau)\,\Gamma(u_x, u_x{}^t; t-\tau)\,dt. \end{aligned}$$

Using (A.3) and (2.7), we can easily show that the $| \cdot |_{0,\infty}$ norms of this Volterra operator is less than $\mathscr{R}$ which is less than 1. As a consequence, the lemma follows.

Proof of Theorem 4.1. If $\varphi \in C^2([0, \pi] \times [0, \infty))$ and vanishes at 0 and π for every $t \geqslant 0$, then it readily follows that

$$\begin{aligned} \| \varphi \| (t) &\leqslant \pi^{1/2} | \varphi | (t) \leqslant \pi \| \varphi \|_x (t) \leqslant \pi^{3/2} | \varphi_x | (t) \\ &\leqslant \pi^2 \| \varphi_{xx} \| (t) \leqslant \pi^{5/2} | \varphi_{xx} | (t). \end{aligned}$$

This fact together with Lemmas 4.2 and 4.3 imply the theorem.

It is of interest to compare this theorem and the development of the corresponding a priori estimates of [2]. The proof of Lemma 4.2 using Fourier series methods differs greatly from the proof of the analogous estimates in [2]. The approach in [2] to the other estimates of this theorem is basically the same as that of Lemmas 4.1 and 4.3.

Having Theorem 4.1 we are able to construct a contractive mapping scheme similar to that of [2]. As a consequence we are able to show that there exists a unique solution of problem (P_λ), $\lambda > 0$. We refer the reader to [1] for details.

5. Asymptotic Stability

In this section we shall prove that $\| u_t \| (t)$ and $| u_t | (t)$ approach 0 as t approaches ∞. Since

$$\int_0^{t_2} \| u_t \|^2 (t)\, dt \leqslant \pi^2 \int_0^{t_2} \| u_{tx} \|^2 (t)\, dt,$$

(4.2) implies that $\| u_t \|^2 (t)$ is integrable on $(0, \infty)$. Thus, $\| u_t \| (t) \to 0$ follows if we can show uniform continuity on $(0, \infty)$. Multiplication of (1.1) by u_t, integration over $[0, \pi] \times [t_1, t_2]$, integrations by parts and the use of (1.3) yield

$$\begin{aligned} \tfrac{1}{2} (\| u_t \|^2 (t_2) - \| u_t \|^2 (t_1)) = &- \int_{t_1}^{t_2} \int_0^{\pi} u_{tx} \sigma(u_x, u_x{}^t)\, dx\, dt \\ &- \lambda \int_{t_1}^{t_2} \int_0^{\pi} u_{tx}^2\, dx\, dt + \int_{t_1}^{t_2} \int_0^{\pi} u_t f\, dx\, dt. \end{aligned} \tag{5.1}$$

Uniform continuity follows readily from (5.1), Theorem 4.1 and repeated use of Schwarz's inequality. Letting

$$a_n(t) = \frac{2}{\pi} \int_0^{\pi} u_t(x, t) \sin nx\, dx, \qquad n = 1, 2, \ldots,$$

we have that

$$\lim_{t \to \infty} \| u_t \|^2 (t) = \lim_{t \to \infty} \sum_{n=1}^{\infty} a_n{}^2(t) = 0.$$

Hence,

$$\lim_{t \to \infty} a_n(t) = 0 \tag{5.2}$$

uniformly with respect to n.

Recall that (1.1) leads to a family of ordinary differential equations in $a_n(t)$ which have as solutions

$$a_n(t_2) = \int_0^{t_2} \exp[- \lambda n^2(t_2 - t)]\, F_n(t)\, dt, \tag{5.3}$$

where

$$F_n(t) = \int_0^\pi \sin nx\, f(x, t)\, dx + \int_0^\pi \sin nx \frac{\partial}{\partial x} \sigma(u_x, u_x^{\,t})\, dx.$$

Using (A.3), (A.5), (3.1), Theorem 4.1 and Schwarz's inequality, we see that, for all $t \in R^+$,

$$F_n(t) \leqslant C_2 \left(|f|_\infty, \int_0^\infty \int_0^\pi f^2, M_0, N_0, \int_0^\infty h \right).$$

Thus, (5.3) implies

$$|a_n(t)| \leqslant \frac{C_2}{\lambda n^2}. \tag{5.4}$$

By the Weirstrass M-test, (5.4) implies that $\sum_{n=1}^\infty a_n(t) \sin nx$ converges uniformly. This fact together with (5.2) implies that

$$\lim_{t\to\infty} |u_t|(t) = 0.$$

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